

$$\int S = \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial \bar{q}} - \frac{d}{dt} \frac{\partial L}{\partial \dot{\bar{q}}} \right\} \delta \bar{q}_{11}$$

$$\Rightarrow \{ \dots \} \perp \text{const. surf.} \Rightarrow \{ \dots \} = \sum_i \lambda_i \frac{\partial G_i}{\partial \bar{q}}$$

$$L \rightarrow L' = L - \sum_i \lambda_i G_i$$

$$\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j}, \quad \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{q}_j} \right) = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right)$$

$$\frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial q_j} - \sum_i \lambda_i \frac{\partial G_i}{\partial q_j}$$

$$\Rightarrow \frac{\partial L'}{\partial q_j} - \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{q}_j} \right) = \frac{\partial L}{\partial q_j} - \sum_i \lambda_i \frac{\partial G_i}{\partial q_j} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_j} \right) = 0$$

$$\frac{\partial L'}{\partial \lambda_i} = 0, \quad \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{\lambda}_i} \right) = 0, \quad \frac{\partial L'}{\partial \lambda_i} = -G_i, \quad G_i = 0$$

$$\Rightarrow \frac{\partial L'}{\partial \lambda_i} - \frac{d}{dt} \left( \frac{\partial L'}{\partial \dot{\lambda}_i} \right) = 0$$

# Hamilton Formalism

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \quad \begin{array}{l} \text{canonically} \\ \text{conjugate} \\ \text{momentum} \end{array}$$

(generalized momentum)

We had:  $\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}$

$$\frac{d}{dt} p_i = \dot{p}_i = \frac{\partial L}{\partial q_i} \quad p_i = p_i(\{q_i\}, \{\dot{q}_i\})$$

2nd order in time,  $f$  equations ( $f$ : # of degrees of freedom)

→ map this out to a system of  $2f$  eq. of motion,  
all 1st order in time

like:  $\ddot{x} = f(x) \Leftrightarrow \begin{cases} \dot{x} = v \\ \dot{v} = f(x) \end{cases}$

BUT: Replace  $q_i$  by  $p_i$  How?

Remark: Dimension  $(q_i, p_i) = \int ds = \text{Dimension (action)}$   
regardless of the nature of the  $q_i$ !

Space of the  $\{q_i\}, \{p_i\} \equiv \text{PHASE SPACE}$

Legendre transformation!

$L$ : "natural variables":  $q_i, \dot{q}_i$

$$dL = \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \sum_i \frac{\partial L}{\partial q_i} dq_i = \sum_i \underline{p_i} d\underline{\dot{q}_i} + \sum_i \frac{\partial L}{\partial q_i} \underline{dq_i}$$

We want a function  $H$  whose natural variables

$$q_i, p_i \Rightarrow dH = \sum_i \dots d p_i + \sum_i \dots d q_i$$

Observation:  $d(p_i \dot{q}_i) = \dot{q}_i dp_i + p_i d\dot{q}_i$

$$p_i d\dot{q}_i = -\dot{q}_i dp_i + d(p_i \dot{q}_i)$$

$$dL = \sum_i \frac{\partial L}{\partial q_i} dq_i - \sum_i \dot{q}_i dp_i + \sum_i d(p_i \dot{q}_i)$$

or

$$d(\sum_i p_i \dot{q}_i - L) = \sum_i \dot{q}_i dp_i - \sum_i \frac{\partial L}{\partial q_i} dq_i$$

Define:  $H = \sum_i p_i \dot{q}_i - L$  Hamilton function

$$H = H(\{q_i\}, \{p_i\})$$

$$dH = \sum_i \dot{q}_i dp_i - \sum_i \frac{\partial L}{\partial q_i} dq_i \Rightarrow$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$-\frac{\partial L}{\partial q_i} = \frac{\partial H}{\partial q_i}$$

$$\text{eq of motion} \Rightarrow \frac{\partial L}{\partial q_i} = \dot{p}_i \Rightarrow \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

$$\Rightarrow \left[ \begin{array}{l} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{array} \right]$$

Hamilton's equation  
of motion

example: harmonic oscillator

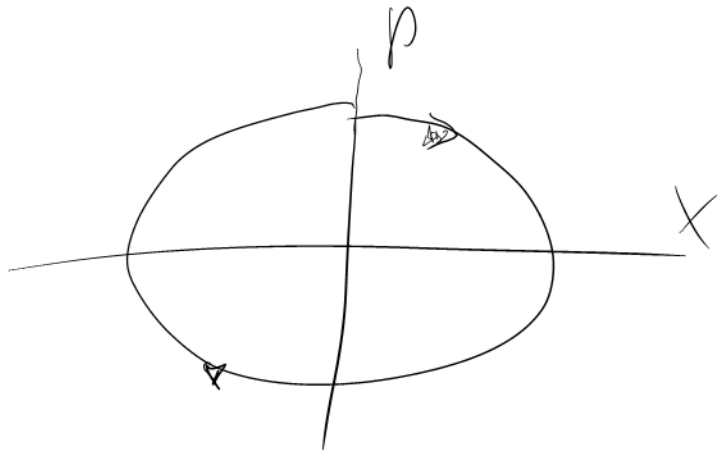
$$L = \frac{m}{2} \dot{x}^2 - \frac{1}{2} m \omega^2 x^2 \quad p = \frac{\partial L}{\partial \dot{x}} = m \dot{x}$$

$$\dot{x} = \frac{1}{m} p \Rightarrow L = \frac{m}{2} \frac{1}{m^2} p^2 - \frac{1}{2} m \omega^2 x^2 = \frac{p^2}{2m} - \frac{m \omega^2}{2} x^2$$

$$H = p\dot{x} - \mathcal{L} = p \frac{1}{m} p - \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2$$

$$= \frac{p^2}{2m} + \frac{m\omega^2}{2} x^2 = \underline{E} \quad \text{energy conserved}$$

ellipsoid



"energy shell"

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

$$\ddot{x} = \frac{1}{m} \dot{p} = \frac{1}{m} (-m\omega^2 x) = -\omega^2 x$$

# Hamilton's eq. of motion via variational principle

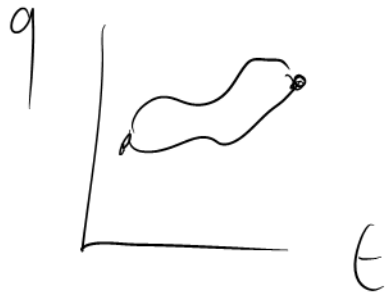
---

$$S = \int_{t_1}^{t_2} dt \mathcal{L} = \int_{t_1}^{t_2} dt (\sum_i p_i \dot{q}_i - H) = S[\vec{q}(t), \vec{p}(t)]$$

↳ depends on the trajectory

↳  $q_i$  ( $\{p_i\}, \{q_i\}$ )

in phase space  $\delta q_i = 0$  for  $t = t_1, t = t_2$



NO constraints / conditions on  $p_i$

$$SS = \int_{t_1}^{t_2} dt \left\{ \sum_i p_i \delta q_i + \sum_i \dot{q}_i \delta p_i - \sum_i \frac{\partial H}{\partial q_i} \delta q_i - \sum_i \frac{\partial H}{\partial p_i} \delta p_i \right\}$$

partial  
integr.

$$\int_{t_1}^{t_2} dt \left\{ \underbrace{- \sum_i \dot{p}_i \delta q_i}_{\text{wavy}} + \underbrace{\sum_i \dot{q}_i \delta p_i}_{\text{dashed}} - \sum_i \frac{\partial H}{\partial q_i} \delta q_i - \sum_i \frac{\partial H}{\partial p_i} \delta p_i \right\}$$

$\delta q_i, \delta p_i$  independent

$$\Rightarrow \dot{p}_i + \frac{\partial H}{\partial q_i} = 0, \quad \dot{q}_i - \frac{\partial H}{\partial p_i} = 0 \quad \checkmark$$

# Poisson Brackets

---

Observable  $f \equiv$  dynamic quantity = function of the phase space variables  $f = f(\{q_i\}, \{p_i\}, t)$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_i \frac{\partial f}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial f}{\partial p_i} \dot{p}_i =$$

$$= \frac{\partial f}{\partial t} + \sum_i \left( \frac{\partial H}{\partial p_i} \frac{\partial f}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial f}{\partial p_i} \right)$$

Poisson

Define:  $\{A, B\} := \sum_i \left( \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} \right)$  bracket

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \{H, f\} \quad \text{general eq. of motion}$$

properties:  $\{A, B\} = -\{B, A\} \Rightarrow \{A, A\} = 0$

elementary PB:  $\{q_i, q_j\} = 0 = \{p_i, p_j\}$

$$\{p_i, q_j\} = \delta_{ij}$$

product rule:  $\{A, B, C\} = A\{B, C\} + \{A, C\}B$

# Canonical Transformations

$$\{p_i\}, \{q_i\} \rightarrow \{P_i\}, \{Q_i\}$$

(transformation in phase space) is called canonical if

$$\dot{P}_i = - \frac{\partial H}{\partial Q_i} \quad \dot{Q}_i = \frac{\partial H}{\partial P_i}$$

$$H = H(\{P_i\}, \{Q_i\})$$

for any Hamiltonian

Check it out!

$$\dot{Q}_i = \underbrace{\frac{\partial Q_i}{\partial t}}_{=0} + \{H, Q_i\} = \{H, Q_i\}$$

$$\dot{P}_i = \{H, P_i\}$$

$$\{H, Q_i\} = \sum_j \left( \frac{\partial H}{\partial p_j} \frac{\partial Q_i}{\partial q_j} - \frac{\partial H}{\partial q_j} \frac{\partial Q_i}{\partial p_j} \right)$$

$$\frac{\partial H}{\partial p_j} = \frac{\partial}{\partial p_j} H(\{Q_\mu\}, \{P_\mu\}) = \sum_\mu \left( \frac{\partial H}{\partial P_\mu} \frac{\partial P_\mu}{\partial p_j} + \frac{\partial H}{\partial Q_\mu} \frac{\partial Q_\mu}{\partial p_j} \right)$$

$$\frac{\partial H}{\partial q_j} = \sum_\mu \left( \frac{\partial H}{\partial P_\mu} \frac{\partial P_\mu}{\partial q_j} + \frac{\partial H}{\partial Q_\mu} \frac{\partial Q_\mu}{\partial q_j} \right)$$

$$\{H, q_i\} = \sum_{j \neq i} \frac{\partial H}{\partial p_j} \left( \frac{\partial p_j}{\partial p_k} \frac{\partial q_i}{\partial q_j} - \frac{\partial p_j}{\partial q_j} \frac{\partial q_i}{\partial p_j} \right)$$

$$+ \sum_{j \neq i} \frac{\partial H}{\partial q_j} \left( \frac{\partial q_j}{\partial p_j} \frac{\partial q_i}{\partial q_j} - \frac{\partial q_j}{\partial q_j} \frac{\partial q_i}{\partial p_j} \right) =$$

$$= \sum_k \frac{\partial H}{\partial p_k} \{p_k, q_i\} + \sum_k \frac{\partial H}{\partial q_k} \{q_k, q_i\}$$

$$\stackrel{!}{=} \dot{q}_i = \frac{\partial H}{\partial p_i} \Rightarrow \{p_k, q_i\} = \delta_{ki}, \quad \{q_k, q_i\} = 0$$

$\{H, P_i\}$  : analogously

=> fundamental Poisson brackets must  
also hold for  $\{P_i\}, \{Q_i\}$

(necessary & sufficient for "canonicity")

---

THEOREM  $\{q_i\}, \{p_i\} \rightarrow \{Q_i\}, \{P_i\}$  canonical  
tr.

$$\{A, B\}_{p, q} = \sum_i \left( \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} - \frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} \right)$$

$$\{A, B\}_{P, Q} = \sum_i \left( \frac{\partial A}{\partial P_i} \frac{\partial B}{\partial Q_i} - \frac{\partial A}{\partial Q_i} \frac{\partial B}{\partial P_i} \right)$$

$$\underline{\text{Thm}} \quad \{A, B\}_{p, q} = \{A, B\}_{P, Q}$$

Proof: Interpret  $A$  as a Hamiltonian!

$$\frac{d}{dt} B = \frac{\partial B}{\partial t} + \{A, B\}_{p, q} \quad \text{in } p, q\text{-reps}$$

$$\frac{d}{dt} B = \frac{\partial B}{\partial t} + \{A, B\}_{P, Q} \quad \text{in } P, Q\text{-reps.}$$

these must be identical!!